Containment Properties of Product and Power Graphs

Antonio Fernández [a] Tom Leighton [b] José Luis López-Presa [c]

[a] *ESCET, Universidad Rey Juan Carlos, Spain, afernandez@acm.org*

[b] *Department of Mathematics and Lab. for Computer Science, MIT, USA, ftl@math.mit.edu*

[c] *DIATEL, Universidad Politécnica de Madrid, Spain, jllopez@diatel.upm.es*

Abstract

In this paper we study containment properties of graphs in relation with the Cartesian product operation. We show that the isomorphism of two Cartesian powers G^r and H^r implies the isomorphism of G and H, while $G^r \subseteq H^r$ does not imply $G \subseteq H$, even for the special cases when G and H are connected or have the same number of nodes. Then, we find a simple sufficient condition under which the containment of products implies the containment of the factors: if $\prod_{i=1}^{n} G_i \subseteq \prod_{j=1}^{n} H_j$, where all graphs G_i are connected and no graph H_j has 4-cycles, then each G_i is a subgraph of a different graph H_j. Hence, if G is connected and H has no 4-cycles, then $G^r \subseteq H^r$ implies $G \subseteq H$. These results can be used to derive embedding results for interconnection networks for parallel architectures.

Key words: Cartesian product, subgraph, isomorphism, embedding, product graphs, power graphs, interconnection networks.

1 Introduction

The Cartesian product has been found as an useful tool to build large graphs from small factor graphs. For instance, there has been a number of interconnection networks proposed for parallel architectures that are, in fact, the Cartesian product of factor networks (e.g., [1,3,5]). Part of the interest of this class of networks is that many of their properties can be derived from the properties of the factor networks [2,7].

A very important property of an interconnection network is its capability of emulating other networks via embeddings. It is well known that the embedding properties of the factor networks propagate to the product network [4, pg.

401][7]. For instance, if G can be efficiently embedded into H, then G^r (the rth Cartesian power of G) can be embedded into H^r with the same efficiency. However, to our knowledge, it is not known whether embedding properties of the product network imply similar embedding properties for the factor networks.

In this work we start looking at this open question by considering containment between graphs, the simplest kind of embedding. Hence, the question we try to answer is the following: "given that one product graph is a subgraph of another product graph, what can we say about the factor graphs?" In the particular case of product interconnection networks, answers to this question would allow to know whether two networks can be subgraph one of the other by only looking at their respective factor graphs.

We first look at power graphs, and show that, if G^r and H^r are isomorphic, then G and H are also isomorphic. This result could drive to conjecture that, if G^r is a subgraph of H^r, then G must be a subgraph of H. However, we disprove this conjecture by presenting counterexamples, even for the special cases when G and H are connected or have the same number of nodes.

We then present a sufficient condition under which the containment of product graphs implies the containment of the factor graphs. We show that, if the product of n *connected* graphs $G_1, ..., G_n$ is a subgraph of the product of n graphs $H_1, ..., H_n$ *without 4-cycles,* then each graph G_i is a subgraph of a different graph H_j. As a consequence, applying this result to power graphs, if G is connected and H has no 4-cycles, then $G^r \subseteq H^r$ implies $G \subseteq H$.

Definition 1 *The* Cartesian product *of two factor graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is the graph $G \times H$ whose vertex set is $V_G \times V_H$ and whose edge set contains all the edges $(uv, u'v')$ such that $\{u, u'\} \subseteq V_G$, $\{v, v'\} \subseteq V_H$, and either $u = u'$ and $(v, v') \in E_H$, or $v = v'$ and $(u, u') \in E_G$.*

2 Containment of Power Graphs

As we said above, we first study the containment properties of power graphs. Our first result has to do with isomorphic power graphs.

Theorem 1 *$G^r = H^r$ implies $G = H$.*

When graphs G and H are connected, the statement of the theorem can be easily shown from the fact that any connected (finite unlabeled) graph has a unique decomposition into a collection of prime factor graphs, which was shown by Sabidusi in [6]. However, Theorem 1 holds for graphs G (and H)

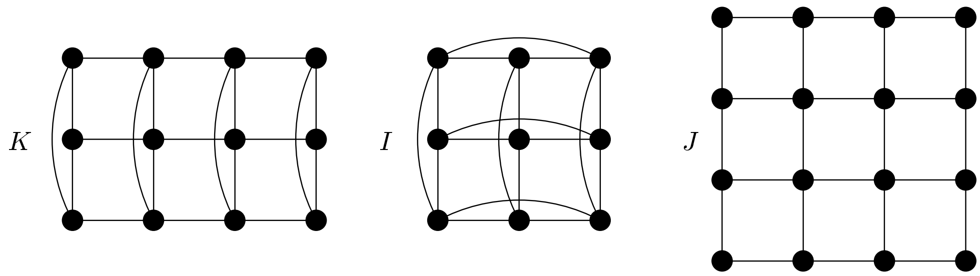

Fig. 1. Graphs K, I, and J.

with any number of connected components, and the proof for the general case is quite more involved.

The above theorem could lead us to conjecture that a similar result holds when G^r and H^r are not isomorphic, but subgraphs one of the other. The following theorem disproves this conjecture.

Theorem 2 *$G^r \subseteq H^r$ does not imply $G \subseteq H$.*

Proof: To prove the theorem we find two graphs G and H such that G is not a subgraph of H but G^2 is a subgraph of H^2. Let us consider the graphs K, I, and J, presented in Figure 1. Clearly, K is not a subgraph of J, since J does not contain a 3-cycle as a subgraph. Similarly, K is not a subgraph of I since K has 12 nodes while I only has 9.

However, observe that K^2 is isomorphic to $J \times I$. Then, if we define $G = K$ and H as the graph with components J and I, we have two graphs G and H such that G^2 is contained in H^2 but G is not a subgraph of H. ■

Note that the counterexample we just presented in the above proof does not cover two large special classes of graphs: connected graphs and graphs with the same number of nodes. Counterexamples for these special cases can be obtained by slightly modifying the one above.

3 Containment of Products of Graphs

In this section we give a sufficient condition for a collection of factor graphs $G_1, ..., G_n$ to be subgraph of another collection $H_1, ..., H_n$ when their respective products are.

Theorem 3 *Let $G_1, ..., G_n$ be n connected graphs with at least one edge and let $H_1, ..., H_n$ be n graphs without 4-cycles. Then, $\prod_{i=1}^{n} G_i \subseteq \prod_{j=1}^{n} H_j$ implies that $(\forall i \in \{1, ..., n\}, \exists k_i \in \{1, ..., n\} : G_i \subseteq H_{k_i})$ and $(\forall i, j \in \{1, ..., n\}, i \neq$*

$j \Rightarrow k_i \neq k_j$), i.e. each graph G_i is a subgraph of a different graph H_j.

This result trivially applies to power graphs: if G is connected and H does not have 4-cycles, then $G^r \subseteq H^r$ implies $G \subseteq H$. Thus, we have found a sufficient condition for this last statement to hold.

Note that a number of graphs used as factors to construct interconnection networks have no 4-cycles (except in specific instances), e.g. the linear array, the ring, any tree, the cube-connected cycles, the mesh of trees [4], or the Petersen graph [5]. Hence, these results can be applied to products or powers of these graphs.

4 Conclusions

We study the containment properties of factor graphs given the containment of product and power graphs, presenting positive and negative results. There are several interesting open questions, like finding simpler sufficient conditions for containment than the one described in Theorem 3.

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